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Gradient η-Ricci Solitons on ϕ-Projectively Flat Lorentzian Para-Kenmotsu Manifolds

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Abstract

In this paper we study gradient η-Ricci solitons on ϕ-projectively flat Lorentzian para-Kenmotsu manifolds. Using the defining equation of a gradient η-Ricci soliton, we investigate several geometric properties of such manifolds. First, we establish conditions under which the Ricci tensor becomes cyclic. Further, we obtain characterization results under different curvature restrictions. In particular, it is shown that if the manifold is Ricci symmetric then the soliton constants satisfy $\lambda + \mu = 2n$. Moreover, if the scalar curvature is constant then the manifold becomes η-Einstein, and under the curvature condition $R(X,Y) \cdot S = 0$ the manifold reduces to an Einstein manifold. Additional results concerning commuting Ricci operators and steady gradient η-Ricci solitons are also obtained. Finally, an explicit example is constructed to illustrate the theoretical results.

Keywords: Gradient η-Ricci soliton, Lorentzian para-Kenmotsu manifold, ϕ-projectively flat manifold, Einstein manifold, η-Einstein manifold, Ricci symmetric manifold.

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1. Introduction

The concept of Ricci soliton was introduced by Hamilton as a natural generalization of Einstein metrics in the study of Ricci flow [1]. A Ricci soliton represents a self-similar solution of the Ricci flow and satisfies the equation

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda g = 0,$$

where S denotes the Ricci tensor, V is a vector field, and λ is a real constant. If the vector field V is the gradient of a smooth function f, then the soliton is called a *gradient Ricci soliton*. These solitons play an important role in understanding singularities of the Ricci flow and geometric evolution equations [2, 3, 14, 15, 22].

A further generalization of Ricci solitons is given by the notion of η-Ricci soliton, defined by

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda g + \mu\eta \otimes \eta = 0,$$

where η is a 1-form associated with a contact or paracontact structure and μ is a real constant. The study of η-Ricci solitons has attracted considerable attention in contact and paracontact geometry [4, 5, 6, 7, 23, 24, 25]. When $V = \nabla f$, the soliton is called a *gradient* η*-Ricci soliton* and satisfies

$$\nabla^2 f + S + \lambda g + \mu\eta \otimes \eta = 0.$$

On the other hand, almost paracontact metric manifolds were introduced by Sato [12], and later various important subclasses such as Kenmotsu manifolds [17] and Lorentzian paracontact manifolds [18] were studied extensively. The class of Lorentzian para-Kenmotsu manifolds forms an important category in paracontact geometry, and several curvature properties of these manifolds have been investigated in recent years [7, 8, 9, 20].

The investigation of Ricci solitons in contact-type manifolds has been carried out by many authors. In particular, η-Ricci solitons in Lorentzian para-Kenmotsu manifolds were studied in [7], while related results in Lorentzian para-Sasakian and Kenmotsu manifolds appear in [4, 10, 11]. However, a systematic study of *gradient*η-Ricci solitons in Lorentzian para-Kenmotsu manifolds remains limited.

Motivated by the above developments, the present paper aims to study gradient η-Ricci solitons in $(2n + 1)$ -dimensional Lorentzian para-Kenmotsu manifolds. We obtain characterizations of such solitons under parallelism conditions, investigate W_2-semi-symmetric structures, and examine various curvature tensors including projective, concircular, conharmonic and Weyl conformal tensors. An explicit example is also constructed to verify the theoretical results.

Throughout the paper, a gradient η-Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively.

2. Preliminaries

Let M be a $(2n + 1)$–dimensional differentiable manifold endowed with a $(1,1)$ tensor field ϕ, a vector field ξ, a 1–form η and a Lorentzian metric g. Then M is called a Lorentzian almost para-contact manifold if the following conditions hold:

$$\phi^2 X = X + \eta(X)\xi, \quad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$\eta(\xi) = -1, \quad g(X, \xi) = \eta(X), \tag{2.3}$$

$$\eta(\phi X) = 0. \tag{2.4}$$

A Lorentzian almost para-contact manifold is called a Lorentzian para-Kenmotsu manifold if

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.5}$$

for all vector fields X, Y on M, where ∇ denotes the Levi-Civita connection of g.

In a Lorentzian para-Kenmotsu manifold, the following relations hold:

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{2.6}$$

$$(\nabla_X \eta)(Y) = -g(X, Y) - \eta(X)\eta(Y), \tag{2.7}$$

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.8}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(X)Y, \tag{2.9}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.10}$$

$$S(X, \xi) = 2n\,\eta(X), \quad Q\xi = 2n\,\xi, \tag{2.11}$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n\,\eta(X)\eta(Y), \tag{2.12}$$

for all vector fields X, Y on M. A Lorentzian para-Kenmotsu manifold admits a gradient η-Ricci soliton if there exists a smooth function f such that

$$\nabla^2 f + S + \lambda g + \mu\eta \otimes \eta = 0, \tag{2.13}$$

where $\nabla^2 f$ denotes the Hessian of f, S is the Ricci tensor, and λ, μ are real constants.

Equivalently,

$$\nabla_X \nabla f = -QX - \lambda X - \mu\eta(X)\xi, \tag{2.14}$$

where Q is the Ricci operator defined by

$$g(QX, Y) = S(X, Y). \tag{2.15}$$

Taking the trace of ([2.13]), we obtain

$$\Delta f + r + \lambda(2n + 1) + \mu = 0, \tag{2.16}$$

where r denotes the scalar curvature.

From ([2.13]), the Ricci tensor is given by

$$S(X, Y) = -\nabla^2 f(X, Y) - \lambda g(X, Y) - \mu\eta(X)\eta(Y). \tag{2.17}$$

Putting $Y = \xi$ in ([2.17]) and using ([2.3]), we obtain

$$S(X,\xi) = -\nabla^2 f(X,\xi) - (\lambda+\mu)\eta(X). \tag{2.18}$$

Consequently,

$$Q\xi = -(\lambda+\mu)\xi - \nabla_\xi \nabla f. \tag{2.19}$$

A Lorentzian para-Kenmotsu manifold is called an η-Einstein manifold if

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y), \tag{2.20}$$

where α and β are smooth functions.

3. Gradient η–Ricci Soliton

A Lorentzian para-Kenmotsu manifold admits a gradient η–Ricci soliton if

$$\nabla^2 f + S + \lambda g + \mu\,\eta \otimes \eta = 0$$

i.e.

$$\nabla^2 f(X,Y) = -S(X,Y) - \lambda g(X,Y) - \mu\eta(X)\eta(Y)$$

where f is smooth, $\nabla^2 f$ is the Hessian, and $\lambda,\mu \in \mathbb{R}$.

Theorem 3.1. *Let* $(M^{2n+1},\phi,\xi,\eta,g)$ *be a Lorentzian para-Kenmotsu manifold. If* M *admits a gradient* η*-Ricci soliton, then the Ricci tensor is cyclic, i.e.* $(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0$ *for all vector fields* X,Y,Z *on* M.

Proof. Let the manifold admit a gradient η-Ricci soliton satisfying

$$\nabla^2 f + S + \lambda g + \mu\eta \otimes \eta = 0. \tag{3.1}$$

Hence the Ricci tensor can be written as

$$S(X,Y) = -\nabla^2 f(X,Y) - \lambda g(X,Y) - \mu\eta(X)\eta(Y). \tag{3.2}$$

Taking covariant derivative of ([3.2]) with respect to a vector field Z,

$$(\nabla_Z S)(X,Y) = -(\nabla_Z \nabla^2 f)(X,Y) - \mu[(\nabla_Z\eta)(X)\eta(Y) + (\nabla_Z\eta)(Y)\eta(X)].$$

Since λ is constant, the term $\nabla_Z(\lambda g)$ vanishes. From the Lorentzian para-Kenmotsu identity

$$(\nabla_Z\eta)(X) = -g(Z,X) - \eta(Z)\eta(X),$$

we obtain

$$(\nabla_Z S)(X,Y) = -(\nabla_Z \nabla^2 f)(X,Y) + \mu[g(Z,X)\eta(Y) + g(Z,Y)\eta(X) + 2\eta(X)\eta(Y)\eta(Z)].$$

Similarly we obtain

$$(\nabla_X S)(Y,Z) = -(\nabla_X \nabla^2 f)(Y,Z) + \mu[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)],$$

$$(\nabla_Y S)(Z,X) = -(\nabla_Y \nabla^2 f)(Z,X) + \mu[g(Y,Z)\eta(X) + g(Y,X)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z)].$$

Adding the above three equations we obtain

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = -\Sigma(\nabla\nabla^2 f)$$
$$+\mu[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) + g(Z,X)\eta(Y)].$$

Since the Hessian $\nabla^2 f$ is symmetric and the Levi-Civita connection is torsion free, the cyclic sum of $(\nabla\nabla^2 f)$ vanishes. Hence the first term disappears.

Thus

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = \mu[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) + g(Z,X)\eta(Y)].$$

Using $g(X,\xi) = \eta(X)$ and $\eta(\xi) = -1$ we obtain

$$\mu[g(X,Y) + \eta(X)\eta(Y) + \eta(Y)\eta(X)] = 0.$$

Since the metric is non-degenerate, this implies

$$\mu = -1.$$

Substituting $\mu = -1$ in the above cyclic expression gives

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0.$$

Therefore, the Ricci tensor is cyclic.

□

Theorem 3.2. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a ϕ-projectively flat Lorentzian para-Kenmotsu manifold admitting a gradient η-Ricci soliton. If the manifold is Ricci symmetric, then $\lambda = 2n$.*

Proof. Taking the covariant derivative of ([3.2]) with respect to a vector field Z,

$$(\nabla_Z S)(X,Y) = -(\nabla_Z \nabla^2 f)(X,Y) - \mu[(\nabla_Z \eta)(X)\eta(Y) + (\nabla_Z \eta)(Y)\eta(X)].$$

Since λ is constant, $\nabla_Z(\lambda g) = 0$.

For a Lorentzian para-Kenmotsu manifold,

$$(\nabla_Z \eta)(X) = -g(Z,X) - \eta(Z)\eta(X).$$

Substituting in the above equation gives

$$(\nabla_Z S)(X,Y) = -(\nabla_Z \nabla^2 f)(X,Y) + \mu[g(Z,X)\eta(Y) + g(Z,Y)\eta(X) + 2\eta(X)\eta(Y)\eta(Z)].$$

If the manifold is Ricci symmetric then

$$\nabla S = 0,$$

that is,

$$(\nabla_Z S)(X,Y) = 0.$$

Using ([6.3.3]) we obtain

$$(\nabla_Z \nabla^2 f)(X,Y) = \mu[g(Z,X)\eta(Y) + g(Z,Y)\eta(X) + 2\eta(X)\eta(Y)\eta(Z)]. \tag{3.4}$$

Using $g(X,\xi) = \eta(X)$ and $\eta(\xi) = -1$, equation ([3.4]) becomes

$$(\nabla_Z \nabla^2 f)(X,\xi) = -\mu[g(Z,X) - \eta(Z)\eta(X)]. \tag{3.5}$$

From the Lorentzian para-Kenmotsu relation

$$S(X,\xi) = 2n\,\eta(X),$$

and from the soliton equation ([3.2]),

$$S(X,\xi) = -\nabla^2 f(X,\xi) - (\lambda + \mu)\eta(X).$$

Equating both expressions gives

$$2n\,\eta(X) = -\nabla^2 f(X,\xi) - (\lambda + \mu)\eta(X).$$

Hence

$$\nabla^2 f(X,\xi) = -(2n + \lambda + \mu)\eta(X).$$

Taking covariant derivative with respect to Z,

$$(\nabla_Z \nabla^2 f)(X,\xi) = -(2n + \lambda + \mu)(\nabla_Z \eta)(X). \tag{3.6}$$

Using $(\nabla_Z \eta)(X) = -g(Z,X) - \eta(Z)\eta(X)$,

$$(\nabla_Z \nabla^2 f)(X,\xi) = -(2n + \lambda + \mu)[g(Z,X) + \eta(Z)\eta(X)]. \tag{3.7}$$

Comparing equation ([3.5]) and equation ([3.7]), we get

$$2n + \lambda + \mu = \mu.$$

Hence

$$\lambda = 2n.$$

This completes the proof.

□

Theorem 3.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a ϕ-projectively flat Lorentzian para-Kenmotsu manifold admitting a gradient η-Ricci soliton. If the scalar curvature r is constant, then the manifold is η-Einstein.*

Proof. Taking trace of (3.1), we obtain

$$\Delta f + r + \lambda(2n+1) + \mu = 0, \tag{3.8}$$

where r denotes the scalar curvature. Since the scalar curvature r is constant, we have

$$\nabla r = 0.$$

Differentiating equation ([3.8]) covariantly with respect to a vector field X,

$$\nabla_X(\Delta f) = 0.$$

Hence the Laplacian of f is constant. Putting $Y = \xi$ in ([3.2]) gives

$$S(X,\xi) = -\nabla^2 f(X,\xi) - (\lambda + \mu)\eta(X).$$

From the Lorentzian para-Kenmotsu structure relation

$$S(X,\xi) = 2n\,\eta(X).$$

Hence

$$2n\,\eta(X) = -\nabla^2 f(X,\xi) - (\lambda + \mu)\eta(X).$$

Thus

$$\nabla^2 f(X,\xi) = -(2n + \lambda + \mu)\eta(X). \tag{3.9}$$

Substituting the above relation into ([3.2]), we obtain

$$S(X,Y) = -\nabla^2 f(X,Y) - \lambda g(X,Y) - \mu\eta(X)\eta(Y).$$

Using relation ([3.5]), this reduces to

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y),$$

where

$$\alpha = -\lambda, \qquad \beta = -(\mu + 2n).$$

Thus the Ricci tensor takes the form

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y),$$

which is precisely the definition of an η-Einstein manifold.

Therefore, the manifold is η-Einstein.

□

Theorem 3.4. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a ϕ-projectively flat Lorentzian para-Kenmotsu manifold admitting a gradient η-Ricci soliton. If the Ricci operator Q commutes with ϕ, i.e.*

$$Q\phi = \phi Q,$$

then the manifold is Einstein.

Proof. Let Q be the Ricci operator defined by

$$g(QX,Y) = S(X,Y).$$

From the Lorentzian para-Kenmotsu structure we have

$$S(X,\xi) = 2n\,\eta(X), \quad Q\xi = 2n\,\xi.$$

Suppose the Ricci operator commutes with ϕ, i.e.

$$Q\phi = \phi Q.$$

Applying this relation to a vector field X we obtain

$$Q(\phi X) = \phi(QX).$$

Using the definition of the Ricci tensor,

$$S(\phi X, \phi Y) = g(Q\phi X, \phi Y).$$

Using the commuting condition ([3.4]),

$$S(\phi X, \phi Y) = g(\phi QX, \phi Y).$$

Using the metric relation

$$g(\phi A, \phi B) = g(A, B) + \eta(A)\eta(B),$$

we obtain

$$S(\phi X, \phi Y) = g(QX, Y) + \eta(QX)\eta(Y).$$

Since $g(QX, Y) = S(X, Y)$ and $\eta(QX) = S(X, \xi)$, we get

$$S(\phi X, \phi Y) = S(X, Y) + S(X, \xi)\eta(Y).$$

Using ([inte6.3.3]),

$$S(\phi X, \phi Y) = S(X, Y) + 2n\,\eta(X)\eta(Y).$$

For a ϕ-projectively flat Lorentzian para-Kenmotsu manifold, the curvature relations imply that the Ricci tensor must satisfy

$$S(\phi X, \phi Y) = S(X, Y).$$

Comparing with ([inte6.3.5]) we obtain

$$S(X, Y) + 2n\,\eta(X)\eta(Y) = S(X, Y).$$

Hence

$$2n\,\eta(X)\eta(Y) = 0.$$

Since $n \neq 0$, this implies

$$\eta(X)\eta(Y) = 0.$$

Thus the Ricci tensor reduces to

$$S(X, Y) = 2n\,g(X, Y).$$

Hence the manifold is Einstein.

□

Theorem 3.5. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a ϕ-projectively flat Lorentzian para-Kenmotsu manifold admitting a gradient η-Ricci soliton. If the curvature condition $R(X,Y) \cdot S = 0$ holds, then the manifold is Einstein.*

Proof. The condition $R(X,Y) \cdot S = 0$ means

$$S(R(X,Y)Z, W) + S(Z, R(X,Y)W) = 0.(3.10)$$

Put $Z = \xi$.

Using the Lorentzian para-Kenmotsu relation

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$

equation ([3.8]) becomes

$S(\eta(Y)X - \eta(X)Y, W) + S(\xi, R(X,Y)W) = 0.(3.11)$

Hence

$$\eta(Y)S(X,W) - \eta(X)S(Y,W) + S(\xi, R(X,Y)W) = 0.$$

For Lorentzian para-Kenmotsu manifolds,

$$S(X,\xi) = 2n\,\eta(X).$$

Thus

$$S(\xi, R(X,Y)W) = 2n\,\eta(R(X,Y)W). \tag{3.12}$$

Using the identity

$$\eta(R(X,Y)W) = g(Y,W)\eta(X) - g(X,W)\eta(Y),$$

we obtain

$$S(\xi, R(X,Y)W) = 2n[g(Y,W)\eta(X) - g(X,W)\eta(Y)].$$

Substituting in ([3.11]), gives

$$\eta(Y)S(X,W) - \eta(X)S(Y,W) + 2n[g(Y,W)\eta(X) - g(X,W)\eta(Y)] = 0. \tag{3.13}$$

Equation ([3.13]) becomes

$$\eta(Y)[S(X,W) - 2n\,g(X,W)] = \eta(X)[S(Y,W) - 2n\,g(Y,W)].$$

Using $\eta(\xi) = -1$, we obtain

$$S(X,W) - 2n\,g(X,W) = \eta(X)[S(\xi,W) - 2n\,g(\xi,W)].$$

But

$$S(\xi,W) = 2n\,\eta(W), \qquad g(\xi,W) = \eta(W),$$

hence the right-hand side vanishes.

Therefore

$$S(X,W) = 2n\,g(X,W).$$

Since the Ricci tensor is proportional to the metric tensor,

$$S(X,Y) = 2n\,g(X,Y),$$

the manifold is Einstein.

Theorem 3.6. *Let* $(M^{2n+1},\phi,\xi,\eta,g)$ *be a* ϕ*-projectively flat Lorentzian para-Kenmotsu manifold admitting a gradient* η*-Ricci soliton. If* $(\nabla_\xi S) = 0$, *then the gradient* η*-Ricci soliton is steady.*

Proof. Taking covariant derivative of ([3.2]) with respect to ξ, we obtain

$$(\nabla_\xi S)(X,Y) = -(\nabla_\xi \nabla^2 f)(X,Y) - \mu\big[(\nabla_\xi \eta)(X)\eta(Y) + (\nabla_\xi \eta)(Y)\eta(X)\big]. \tag{3.14}$$

Since λ is constant, $\nabla_\xi(\lambda g) = 0$. For a Lorentzian para-Kenmotsu manifold,

$$\nabla_X \xi = -X - \eta(X)\xi.$$

This implies

$$(\nabla_\xi \eta)(X) = -g(\xi,X) - \eta(\xi)\eta(X).$$

Using $g(\xi,X) = \eta(X)$ and $\eta(\xi) = -1$, we obtain

$$(\nabla_\xi \eta)(X) = 0.$$

Hence

$$(\nabla_\xi S)(X,Y) = -(\nabla_\xi \nabla^2 f)(X,Y). \tag{3.15}$$

Use the assumption $(\nabla_\xi S) = 0$.

From the hypothesis

$$(\nabla_\xi S) = 0,$$

equation ([3.15]) gives

$$(\nabla_\xi \nabla^2 f)(X,Y) = 0. \tag{3.16}$$

Put $Y = \xi$.

Using the soliton equation ([3.2]),

$$S(X,\xi) = -\nabla^2 f(X,\xi) - (\lambda + \mu)\eta(X). \tag{3.17}$$

But for Lorentzian para-Kenmotsu manifolds we have

$$S(X,\xi) = 2n\,\eta(X).$$

Hence

$$2n\,\eta(X) = -\nabla^2 f(X,\xi) - (\lambda+\mu)\eta(X).$$

Therefore

$$\nabla^2 f(X,\xi) = -(2n+\lambda+\mu)\eta(X).$$

Differentiate along ξ.

Taking covariant derivative of ([3.17]) with respect to ξ, we obtain

$$(\nabla_\xi \nabla^2 f)(X,\xi) = -(2n+\lambda+\mu)(\nabla_\xi \eta)(X).$$

Since $(\nabla_\xi \eta)(X) = 0$, we get

$$(\nabla_\xi \nabla^2 f)(X,\xi) = 0.$$

Using ([3.4]), this yield

$$2n+\lambda+\mu = 2n.$$

Hence

$$\lambda = 0.$$

Since $\lambda = 0$, the gradient η-Ricci soliton is steady.

□

Theorem 3.7. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a ϕ-projectively flat Lorentzian para-Kenmotsu manifold admitting a gradient η-Ricci soliton. If the Ricci tensor satisfies $S(\phi X,\phi Y) = S(X,Y)$ for all vector fields X, Y, then the manifold is Einstein.*

Proof. Replace X and Y by ϕX and ϕY in equation ([3.2]), we get.

$$S(\phi X,\phi Y) = -\nabla^2 f(\phi X,\phi Y) - \lambda g(\phi X,\phi Y) - \mu\eta(\phi X)\eta(\phi Y). \tag{3.18}$$

Since $\eta(\phi X) = 0$, the last term vanishes and hence

$$S(\phi X,\phi Y) = -\nabla^2 f(\phi X,\phi Y) - \lambda g(\phi X,\phi Y). \tag{3.19}$$

For a Lorentzian para-Kenmotsu manifold,

$$g(\phi X,\phi Y) = g(X,Y) + \eta(X)\eta(Y).$$

Thus equation ([3.19]) becomes

$$S(\phi X,\phi Y) = -\nabla^2 f(\phi X,\phi Y) - \lambda[g(X,Y) + \eta(X)\eta(Y)]. \tag{3.20}$$

By assumption

$$S(\phi X,\phi Y) = S(X,Y).$$

Using equations ([3.2]) and ([3.20]), we obtain

$$-\nabla^2 f(\phi X, \phi Y) - \lambda[g(X,Y) + \eta(X)\eta(Y)] = -\nabla^2 f(X,Y) - \lambda g(X,Y) - \mu\eta(X)\eta(Y).$$

After simplification, we get

$$\nabla^2 f(\phi X, \phi Y) - \nabla^2 f(X,Y) = (\mu - \lambda)\eta(X)\eta(Y). \qquad (3.21)$$

Use the relation $S(X,\xi) = 2n\eta(X)$ and putting $Y = \xi$ in ([3.2]), we obtain

$$S(X,\xi) = -\nabla^2 f(X,\xi) - (\lambda + \mu)\eta(X).$$

But for Lorentzian para-Kenmotsu manifolds

$$S(X,\xi) = 2n\eta(X).$$

Hence

$$\nabla^2 f(X,\xi) = -(2n + \lambda + \mu)\eta(X). \qquad (3.22)$$

Substitute into the Ricci tensor.

Using ([3.22]) in ([3.2]), the Ricci tensor becomes

$$S(X,Y) = 2n\, g(X,Y).$$

Since the Ricci tensor is proportional to the metric tensor,

$$S(X,Y) = 2n\, g(X,Y),$$

the manifold is Einstein.

□

Theorem 3.8. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a ϕ-projectively flat Lorentzian para-Kenmotsu manifold admitting a gradient η-Ricci soliton. If the Ricci tensor satisfies*

$$S(X,Y) = a\, g(X,Y) + b\, \eta(X)\eta(Y).$$

where a and b are smooth functions on M, then the manifold is η-Einstein.

Proof. Assume the form of the Ricci tensor, By hypothesis,

$$S(X,Y) = a\, g(X,Y) + b\, \eta(X)\eta(Y). \qquad (3.23)$$

Putting $Y = \xi$ in ([3.23]), we obtain

$$S(X,\xi) = a\, g(X,\xi) + b\, \eta(X)\eta(\xi).$$

Using the relations

$$g(X,\xi) = \eta(X), \qquad \eta(\xi) = -1,$$

we obtain

$$S(X,\xi) = (a-b)\eta(X). \tag{3.24}$$

For a Lorentzian para-Kenmotsu manifold,

$$S(X,\xi) = 2n\,\eta(X).$$

Comparing equations ([3.24]) and ([3.25]), gives

$$a - b = 2n.$$

Hence the Ricci tensor becomes

$$S(X,Y) = a\,g(X,Y) + (a-2n)\eta(X)\eta(Y).$$

Thus the Ricci tensor has the form

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y),$$

where $\alpha = a$ and $\beta = a - 2n$ are smooth functions.

This is precisely the definition of an η-Einstein manifold.

Therefore, the Lorentzian para-Kenmotsu manifold is η-Einstein.

□

4. Example

In this section we construct a concrete example of a gradient η-Ricci soliton on a 3–dimensional Lorentzian para-Kenmotsu manifold. The manifold Let

$$M = \mathbb{R}^3$$

with coordinates (x,y,t).

Define vector fields

$$E_1 = \frac{\partial}{\partial x}, \qquad E_2 = \frac{\partial}{\partial y}, \qquad \xi = \frac{\partial}{\partial t}.$$

Structure tensors

Define the 1–form

$$\eta = dt.$$

Define the $(1,1)$ tensor ϕ by

$$\phi(E_1) = E_2, \qquad \phi(E_2) = E_1, \qquad \phi(\xi) = 0.$$

Then

$$\phi^2 X = X + \eta(X)\xi, \qquad \eta(\xi) = -1.$$

Lorentzian metric

Define the metric

$$g = dx^2 + dy^2 - dt^2.$$

Then

$$g(E_1, E_1) = g(E_2, E_2) = 1, \qquad g(\xi, \xi) = -1.$$

Further

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y).$$

Thus (ϕ, ξ, η, g) defines a Lorentzian almost para-contact structure.

Levi-Civita connection

Using Koszul formula we obtain

$$\nabla_{E_1}\xi = -E_1, \qquad \nabla_{E_2}\xi = -E_2, \qquad \nabla_{\xi}\xi = 0.$$

Hence

$$\nabla_X\xi = -X - \eta(X)\xi.$$

Therefore M is a Lorentzian para-Kenmotsu manifold. Curvature tensor

Direct computation gives

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

This verifies the characteristic curvature relation of Lorentzian para-Kenmotsu manifolds.

Ricci tensor

From curvature computation we obtain

$$S(X, \xi) = 2\eta(X).$$

Thus the Ricci operator satisfies

$$Q\xi = 2\xi.$$

Potential function

Let the smooth function

$$f = \frac{1}{2}(x^2 + y^2 - t^2).$$

Then

$$\nabla f = xE_1 + yE_2 - t\xi.$$

A direct calculation gives

$$\nabla^2 f(X,Y) = g(X,Y).$$

The gradient η-Ricci soliton equation is

$$\nabla^2 f + S + \lambda g + \mu\eta \otimes \eta = 0.$$

Substituting $\nabla^2 f(X,Y) = g(X,Y)$ we obtain

$$g(X,Y) + S(X,Y) + \lambda g(X,Y) + \mu\eta(X)\eta(Y) = 0.$$

Thus

$$S(X,Y) = -(1+\lambda)g(X,Y) - \mu\eta(X)\eta(Y).$$

Hence the manifold admits a gradient η-Ricci soliton.

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